Mottel's notes on Don Hutton's Moon Calculator

Knowing of my interest in the Moon, Don invited me to offer any improvements I could come up with to his moon calculator. For this, I had to learn not just how to use it, but exactly how it was constructed. Here is my breakdown of the principles involved in its construction. (Note: it is one thing to deconstruct a finished product; it is quite another to design it ab initio. The version I have made is only a slight improvement in construction; the design is the same. But I hope that its by-product – this deconstruction – may help others to a better understanding of the subject.)

1. Terminology and underlying concepts

A lunation (**L**) is the time taken for the Moon to return to a notional starting point (**P**) in its orbit. Several candidates exist for P. Consequently, there are different kinds of lunation, each with its own starting point, and each one has a different length. The length depends on whether P is stationary or moving as seen from our observation point (Earth), and, if moving, its rate and direction of motion relative to the Moon's motion (see Table 1).

A *synodic* lunation is the time taken for the Moon to return to the same position *relative to the Sun* as seen from Earth. It is usually measured from one conjunction (New Moon) to the next[1], but it can also be measured from one opposition (Full Moon) to the next. (The period is the same.) At both points, the Sun, Earth and Moon are in line with one another in the vertical plane (but not always in the horizontal plane[2]). At conjunction, the Moon is in the middle[3] and the Sun and Moon as viewed from Earth both occupy the same celestial longitude (but not necessarily the same latitude). At opposition, the Earth is in the middle and the longitudes occupied by the Sun and Moon as seen from Earth are 180 degrees apart. (Scrabble players please note: At both conjunction and opposition the Moon is said to be in syzygy.)

Neither at conjunction nor at opposition could a single observer on Earth see both bodies at once. At opposition they are on opposite sides of the Earth (though if sunset or sunrise at your location occurs at or near opposition, then with an unobstructed eastern and western horizon, you could witness moonrise at sunset or moonset at sunrise). At conjunction, even if we could somehow filter the Sun's glare completely, we would still not see the Moon because the Moon's near side (the side facing Earth) is its night side, so the Moon will not reflect any sunlight toward Earth.

Table 1: Types of Lunation[4]

Lunar Period [of relevance to]	**Notional Starting Point (P) of orbit** Name	Rate & direction of motion (**m**)	**Duration of Lunation** (**L**), and mean **d**aily rate of motion/elongation[6] (**d** = 360°/L)
Sidereal (r) ["real" orbital period]	Fixed Star	Nil	L_r = **27d 7h 43m 11s** = 27.32165509d d_r = 13° 10' 34.9" = 13.17636134°
Tropical (t) [relates m to d_s, d_n, d_a, – see notes to table]	March Equinox	Precession: (360 x 3600)/(26000 x 12) = m_t ≈ 4" per month ≈ 0.001111111° (opposite)	L_t = [L_r x ((360° − 4")/360°)] = **27d 7h 43m 4s** = 27.32157076d d_t = 13° 10' 35.05" = 13.17640201°
Synodic (s) [Moon's phases]	Centre of Sun	m_s = 59' 8.33" per day = 0.985647222° (same)	L_s = **29d 12h 44m 2s** = 29.53057511d d_s = 12° 11' 26.72" = 12.19075479°
Draconic[5] or Nodal (n) [prediction of eclipses]	Node	m_n = 3' 10.63" per day = 0.052952778° (opposite)	L_n = **27d 5h 5m 35s** = 27.21221146d d_n = 13° 13' 45.68" = 13.22935479°
Anomalistic (a) [apparent size of Moon, hence, types of eclipses]	Lunar Apogee	m_a = 6' 41.10" per day = 0.111416667° (same)	L_a = **27d 13h 18m 35s** = 27.55457176d d_a = 13° 3' 53.95" = 13.06498534°

Notes to Table 1:

L_r and **m** are (presumably) obtained by observation. The other periods appear to have been derived mathematically (though observations of L_s are very old). The derivation of L_t (for which P is the equinox) is shown in the table, and d_t = 360°/L_t, where L_t is in days. (Ditto for all **d**.) Observations of all celestial objects (specifically the Moon and P in this case) are usually referenced to the March equinox ("gamma"). Therefore in principle, the synodic, draconic and anomalistic periods (last 3 rows) are derived from **L** = 360°/**d**, where **d** is the algebraic sum of d_t (the Moon's daily rate of motion[6] in relation to the equinox, 13° 10' 35.05"), and **m** (the starting point's rate of motion). That is to say, **d** is the net result of combining the daily motions of the Moon and P in relation to the equinox. Where the two motions are in the *same* direction, **d** = d_t *minus* **m**; where they are in *opposite* directions, **d** = d_t *plus* **m**. Thus: $d_s = d_t - m_s$, $d_n = d_t + m_n$ and $d_a = d_t - m_a$. From d, we then obtain L by **L** = 360°/**d**. (speed = distance/time, therefore time = distance/speed).

1 Synod means a meeting (in this case of Sun and Moon, when they converge at the same celestial longitude).

2 When the three bodies are aligned in both the vertical plane and the horizontal plane, there is an eclipse.

3 All lunar conjunctions are inferior conjunctions. The inner planets can be seen at either inferior conjunction (Earth, planet, Sun) or superior conjunction (Earth, Sun, planet). All conjunctions of the outer planets with the Sun are superior conjunctions.

4 Table 1 copied with minor emendations and additions from Feldman: *Rabbinical Mathematics and Astronomy*, p 136. (Notes to it adapted from same.) See: http://www.amazon.com/exec/obidos/tg/detail/-/0872030261/102-6192119-0759322?v=glance

5 So named from an old Eastern belief in a dragon in the sky, its head at one node and tail at the other, that eclipsed the Moon by trying to swallow it whenever it approached its head. (Don't know what was supposed to happen at the other end.)

6 See footnote 12.

We are concerned here with mapping the Moon's phases to dates in the Gregorian calendar [7]. Therefore, we are interested here in

the Moon's *synodic* lunation ($\mathbf{L_s}$), which is the time taken for the Moon to complete a full cycle of phases.

The phases are due to the change in the Moon's position relative to both the Earth and the Sun. As the Moon orbits the Earth, the angle Sun, Earth, Moon changes, and with it the proportion of the Moon's near side (the side facing Earth) that is illuminated by the Sun also changes. Since a synodic lunation is measured by the Moon's position *relative to the Sun* (as seen from Earth), it is the synodic lunation that is relevant here. So:

from here on, **L** *(with no subscript indicating the type of lunation) means* $\mathbf{L_s}$*, a synodic lunation.*

The Moon's motion is not uniform. For one thing, its orbit is an ellipse, so its orbital speed is constantly changing. It gets faster the closer it is to perigee (its closest point to Earth) and slower the closer it is to apogee (its furthest point from Earth). Secondly, the Earth's orbit of the Sun is also an ellipse, so its distance from the Sun varies (as does its orbital speed as it approaches perihelion and aphelion). Therefore the force of the Sun's gravitational pull upon the Moon is constantly varying both in magnitude and in direction relative to the Earth's pull upon the Moon. The effect of all this is that some lunations are longer in duration than others.[8] (In addition, the Moon's motion is subject to many minor perturbations from the gravity of other planets, whose influence upon the Moon changes with their configurations relative to the Earth, Moon and Sun.)

Because the Moon's motion is not uniform, we often speak of *mean* lunations and conjunctions. A **mean lunation** is the average time that it takes the Moon to complete a real lunation. **Mean conjunctions** are computed times of lunar conjunctions based on the motions of the *mean Sun* and the *mean Moon*. Both are fictitious bodies to which we ascribe uniform motions that approximate the (apparent) motion of the real Sun and the (real) motion of the real Moon (neither of which is uniform).

A mean *synodic* lunation (**L**) is the interval between one mean lunar *conjunction* and the next. **L** = 29.5d, 44m, 2.761728000s.[9] In days, it is 29.53058752d.

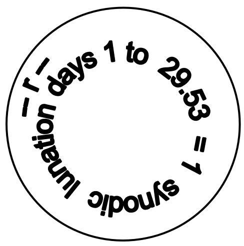
2. Construction of the Moon Calculator

This consists of concentric discs of different diameters with a common axis, around which they can be independently rotated so as to vary the alignment of markings on their rims. Such an instrument is clearly incapable of solving the complex equations needed to predict real conjunctions; it is based on the *mean* synodic lunation and it predicts *mean* conjunctions (and, by extension, other phases). It is worth mentioning here that the time of a real conjunction can differ from a mean conjunction by up to plus or minus 15 hours,[10] though the difference is often considerably less than that. The same of course applies to the other phases.

The first thing to recognise is that (to explicate the […] obvious) the phases and the days-of-month are both arranged on their respective discs in a circle. This is because they both represent a repeating cycle, and since the cycle repeats, if the divisions on the discs are arranged correctly, this instrument becomes a perpetual moon calendar. That cycle is of course a mean synodic lunation (**L**), about 29.53 solar days. Therefore:

a) The Day Disc

The perimeter of this disc must be divided into day segments such that **L** occupies the *full* circle, i.e. the divisions must equal 29.53 days in total. The remaining time (**r**) by which a Gregorian calendar month exceeds L belongs to the next lunation, so r must "overlap" the divisions representing L. (See diagram at right.) The length of **r** is:



1.47d	for a 31-day month	(31 − 29.53 = 1.47),
0.47d	for a 30-day month	(30 − 29.53 = 0.47),
−0.53d	for February of a leap year	(29 − 29.53 = −0.53),
−1.53d	for February of a common year	(28 − 29.53 = −1.53).

That is the general principle. We will now discuss the exact placements of the "extra" days in r.

7 Our present calendar, used throughout the West and in most of the world. Named after Pope Gregory 13th who introduced it in 1582 as a minor reform to the Julian calendar, then in common use. It commenced on Oct 15, 1582.

8 In particular, since the Moon's anomalistic period, L_a, is shorter than its synodic period, L_s, by z days (z ≈ 1.976), if, at [conjunction + s°], the Moon is at [perigee + a°], (0 ≥ s, a < 360), by the next time the Moon is at [conjunction + s°], it will have progressed x° beyond [perigee + a°], where x° is z × (d_a − d_s) ≈ 1.73° (see Table 1 for meanings of d_a and d_s). Moreover, since L_a and L_s have no common factors (because z is so relatively small), it will be a long time before the Moon is again at the same position relative to both the Sun and perigee. Therefore for any s, the Moon's orbital speed at [conjunction + s°] will rarely be the same for two different orbits, so the change in its orbital speed as the Moon nears perigee or apogee will shorten or lengthen the current (true) synodic lunation from its mean value by a different amount each time. (I am indebted to Denis Coates, astronomy lab demonstrator at Monash University, for steering me toward this explanation.)

9 The mean synodic lunation is gradually diminishing at the rate of about 0.33 seconds per millennium. (Feldman p 136, amended to reflect value given in http://individual.utoronto.ca/kalendis/hebrew/Apparent_MSM_vs_hYear.pdf). But from here: http://science.nasa.gov/headlines/y2004/21jul_llr.htm?list503953 it would seem that the Moon is receding from Earth by 3.8 cm per year. See also: http://sunearth.gsfc.nasa.gov/eclipse/SEhelp/ApolloLaser.html . This is puzzling.

10 Feldman pp 146, 185. This is an approximation.

If a mean synodic lunation begins at 11:16 on Jan 1, it will end at 24:00 on Jan 30, which is 0:00 hrs on Jan 31. (At noon on Jan 31 the Moon of the next lunation will be 12 hours old, and the waxing crescent should be visible by sunset[11]). Therefore, days 30 and 31 should be placed on the day disc so that noon of day 1 coincides with the end of day 30 and the beginning of day 31. In other words, "30" should be half a day to the left of "1", and "31" should be half a day to the right of "1". (More precisely, since the lunation began not at noon but 44 minutes earlier, that "half a day" should, in both cases, really be 0.47 of a day, but that is not really practical and the difference is negligible for an instrument like this.)

How far apart should the day divisions on the day disc be drawn? As mentioned above, the full circle (360°) represents one mean synodic lunation (**L**). Therefore a day division represents

the Moon's mean daily elongation ($\mathbf{d_s}$) = 360°/**L** = 360°/29.53058752d = 12.19074968° per day. (This is the average daily increase in the Moon's angular distance from the Sun as seen from Earth.)

This then is the distance in degrees of arc that should separate the day divisions in the drawing. The last division between the end of day 29 and the beginning of day 1 is the amount by which L exceeds 29 whole days. This is 0.53058752d, which, in degrees of arc = (0.53058752d × $\mathbf{d_s}$ =) 6.468259640°.

I want each of the day divisions in this version to be subdivided into quarters. The dividing lines will indicate the times midnight, noon, 6 am and 6 pm. The angular spacing between these subdivisions will be one quarter of the Moon's mean daily elongation: 360°/4L = 3.047687419°.

The degree of precision to which the above values are given is not really crucial for an instrument of this nature. I give these values mainly for expository purposes, though I will use them to this precision in the calculations for the design to avoid compounding of rounding errors and to thereby get the target spacing of the divisions as accurate as possible. But in the actual drawings, the day-divisions will be angled only to the nearest degree of arc. The circles and their divisions will be drawn in Microsoft Word (using drawing objects and WordArt), which only allows lines to be rotated by an integral number of degrees. This is quite satisfactory for an instrument of this nature – as Don himself has pointed out, this calculator is not, indeed cannot be, a precision instrument.

A digression on the Moon's mean daily elongation ($\mathbf{d_s}$)

($\mathbf{d_s}$ = 360°/**L** = 360°/29.53058752d = 12.19074968° per day = 12° 11' 26.7" per day)

From Don's lectures we have become accustomed to associating 13° with the Moon's average daily rate of motion.[12] But that is d_r, the daily motion based on the Moon's *sidereal* period (L_r) [360°/L_r = 360°/27.32165509d = 13.17636134° = 13° 10' 34.9" per day]. Here, we are dealing with the Moon's *phases*, so we are using the Moon's *synodic* lunation ($\mathbf{L_s}$), from which we derive $\mathbf{d_s}$, the Moon's mean daily *elongation*, i.e. the average daily increase in its angular separation *from the Sun* as seen from Earth. (Remember, this is the daily elongation of the *mean* Moon from the *mean* Sun.)

Let us take this digression one step further: The Earth rotates at a speed of 15° per hour (by the Sun – i.e. taking one rotation as being from one upper transit of the Sun (noon) to the next.) This rate = ¼° per min = 1° every four minutes. Multiply the Moon's daily elongation (d_s) by 4 minutes and we get 48.763 (i.e. approx. 48 ¾) minutes, which is the amount of (solar) time that the Moon is later, each day of the lunation, in reaching upper transit (and in rising) than the day before. If it falls behind[13] the Sun by this amount of time each day of the lunation, how much has it fallen behind after completing a full synodic lunation? Don't get out the calculator – the synodic lunation is measured from conjunction to conjunction, so at the end of the lunation the Moon is exactly in line with the Sun again, therefore it will have fallen behind by exactly 24 hours. Now confirm that our calculation of d_s was correct by observing that 48.763 minutes × $\mathbf{L_s}$ = 1440 minutes = 1 day.

11 Common wisdom is that a minimum *true* elongation of 9° is required before the waxing crescent can first be seen. **[**But not always. The closer the Sun and Moon are vertically (i.e. in altitude), the larger the crescent must be for the Sun not to outshine the Moon. The minimum elongation (9°) is sufficient for visibility providing the vertical separation between Sun and Moon is > 14°. At altitude differences between 14° and 9°, the minimum elongation required for visibility increases up to 13° the smaller their altitude difference is. If the altitude difference is < 9° the Sun will outshine the Moon even at much larger elongations (up to ≈ 24°). (Feldman pp 161 & 170.)**]** *Mean* elongation increases at about ½° per hour. Assuming the same rate of increase for *true* elongation (to cover an angular distance of only 9° the difference in time would be negligible), the Moon must be at least 18 hours old to be visible. See http://www.astronomycafe.net/qadir/q947.html and http://www.astronomycafe.net/qadir/q729.html . However, this conjunction is a *mean* conjunction, and the minimum *mean* elongation required for (theoretical) visibility is only 2½° (Feldman, pp 144, 165, 170, 192). Depending on the time difference between this mean conjunction and the true conjunction, the (real) waxing crescent may well be visible by this time.

12 Elongation usually means the angular separation (as viewed from Earth) of a satellite body from its parent body, most commonly a planet from the Sun. We use the term elongation when speaking of the Moon's separation from the Sun (its grandparent body), but when speaking of the Moon's separation from a fixed star or other reference point, it is more correct to use the term "daily rate of motion" rather than elongation.

13 "Behind" in terms of its apparent diurnal motion (East to West) produced by Earth's rotation (in which we clearly see the Moon transiting later and later each day of the lunation). In terms of its monthly (real) motion from West to East of 13° per day against the stars, the *waxing* Moon is said to be ahead of the Sun, which appears to also move Eastward at just under 1° per day (apparent motion produced by Earth's annual orbit of the Sun), and the *waning* Moon is said to be trailing the Sun.

b) The Month Disc

The average Gregorian month length (30.43d) exceeds L (29.53 days) by about 0.9d. Therefore in general, as the year progresses, the Gregorian day-of-month on which a mean lunar conjunction will occur gradually creeps backward by roughly that amount per month. You can see this in Table 2. The first (true) conjunction in 2005 was Jan 10, at 12:03 (UT).[14] The subsequent mean conjunctions for the year (to the nearest minute) are obtained by repeated addition of L to that time. Note the backward creep in the dates.

Table 2: Mean and True[14] Lunar Conjunctions (C) for 2005 to 2006 (Universal Time)

C	Mean	difference	True	C	Mean	difference	True
1			Jan 10 12:03	8	Aug 5 05:11	-02:06 =	Aug 5 03:05
2	Feb 9 00:47	-02:19 =	Feb 8 22:28	9	Sep 3 17:55	+00:50 =	Sep 3 18:45
3	Mar 10 13:31	-04:41 =	Mar 10 09:10	10	Oct 3 06:39	+03:49 =	Oct 3 10:28
4	Apr 9 02:15	-05:43 =	Apr 8 20:32	11	Nov 1 19:23	+06:01 =	Nov 2 01:24
5	May 8 14:59	-06:14 =	May 8 08:45	12	Dec 1 08:08	+06:53 =	Dec 1 15:01
6	Jun 7 03:43	-05:48 =	Jun 6 21:55	13	Dec 30 20:52	+06:20 =	Dec 31 03:12
7	Jul 6 16:27	-04:25 =	July 6 12:02	14	Jan 29 09:36	+04:39 =	Jan 29 14:15

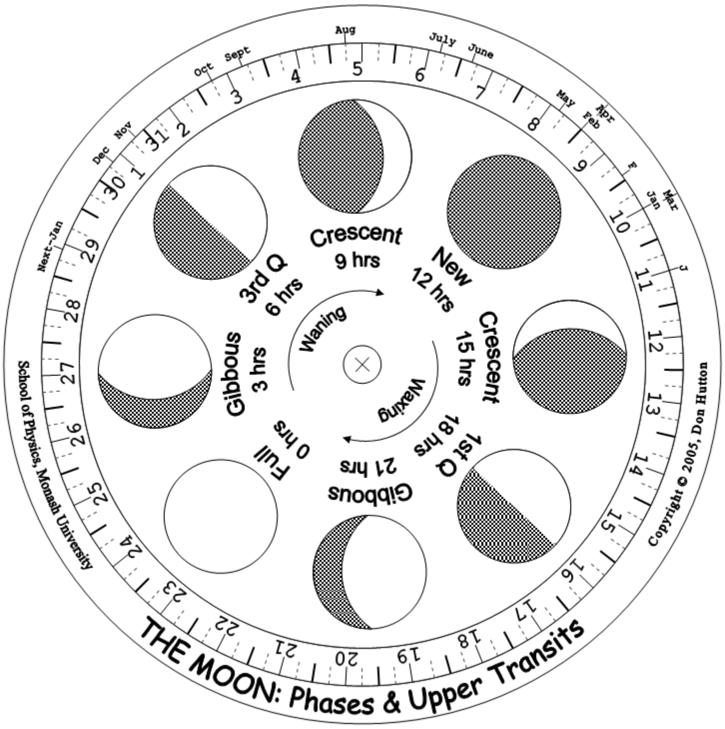
Notice that the date of the 13th *mean* conjunction (ending the 12th lunation) is 11 days short of the anniversary of the first conjunction. This is because 12 mean synodic lunations ≈ 354.37d, which is about 11 days shorter than a calendar year. More precisely, the difference between 12L and a calendar year of either 365 or 366 days is, respectively, 10.63 or 11.63 days. The fraction 0.63 of a day ≈ 15 hours. Therefore, if the first conjunction of the year occurs any time up to January 11 at 15:00 (or in a leap year up to Jan 12 at 15:00), there will be a 13th mean conjunction in December. This too can be seen from the data in Table 2.

For continuity, the month disc also indicates the first conjunction of the following year (Jan at left). This means that you only need a single known (real) conjunction to set up the calculator. After that, it can be used in perpetuity. To reset it for the following year, first observe from Jan at left when the first conjunction of next year will occur (Jan 29 at 09:36 in the above example), then align that date and time with Jan at right and the calculator is now reset for the following year (though with a *mean*, not a true conjunction).

We said just above that 366d − 12L = 11.63d. Now notice that the distance on the month disc between "J" at right and (next) Jan at left is equivalent to 11.63 day divisions on the day disc ($\mathbf{d_s} \times 11.63 \approx 142°$). Now, how precisely should the months on the month disc be positioned relative to one another?

February is shorter than L (29.53d) by 1.53d (common yr) or 0.53d (leap yr). A mean synodic lunation beginning at 0 hrs on Feb 1 will not end until 12:44 on March 2 (common yr) or March 1 (leap yr). Therefore March is to the right of Feb by ($\mathbf{d_s} \times 1.53$)° and to the right of "F" (used instead of Feb in a leap year) by ($\mathbf{d_s} \times 0.53$)°. A 31 or 30-day month is longer than L by 1.47d or 0.47d, respectively. Therefore, each of the other months is shifted left of the previous month by ($\mathbf{d_s} \times 1.47$)° if the previous month has 31 days, or by ($\mathbf{d_s} \times 0.47$)° if the previous month has 30 days. "J" and "F" are right of Jan and Feb by the equivalent of 1 day ($\mathbf{d_s}$°).

In my initial investigation of Don's calculator, I wrote a small program to generate some data (dates and times of mean conjunctions) to test it against. The mean conjunctions in Table 2 is part of the output of that program.[15] In testing this version against the same data, I found that the readings are accurate to within ±3hrs, and often it is within ±2hrs.



At right: Don's Moon Calculator, mark II (my version), set up for April 2005. Date (and UT) of 1st New Moon of year (Jan 10, 12:03) is aligned with Jan at right, and New Moon with month of interest (April). New Moon is on the 9th shortly after midnight and Full Moon on the 23rd shortly before midnight. (Full Moon ≈ New Moon + 14d, 18h, 22m.)

14 Dates and times of true lunar conjunctions are from the U.S. Naval Observatory http://aa.usno.navy.mil

15 That program (source code and .exe) and this document are at http://www.clock.gutnick.net (in MoonCal.zip). For copies of Don Hutton's Moon Calculator, contact the School of Physics, Monash University, Melbourne, Australia.

The version of the Moon Calculator shown above includes instructions for use printed on the back. This is a copy of those instructions. Only the section headed "Limitations" is new. The rest is copied from Don's original instruction sheet with some minor modifications.

Setup

(1) Set date of 1^{st} New Moon of year:
Turn day disc until the correct day (see below)
(at the approx. time) is aligned with Jan˙ at right of month disc.
This sets the dates of all New Moons for the year and for next Jan.

January date and (AEST) time of first New Moon of year:

2005: Jan 10, 22:03	2006: Jan 30, 00:15	2007: Jan 19, 14:01

For more, see: http://aa.usno.navy.mil

(2) Turn phase disc until NEW Moon is against month˙ of interest on month disc.
This sets the day-of-month of all the phases.

- - - - ˙ **In a leap year** use J and F instead of Jan and Feb. **- - - -**

To **View**

Keep all discs fixed and turn the whole instrument until the date or phase of interest is at the top.
Dates (and approximate times) of the Moon's phases* can now be read.
(On the day disc: **━━** = midnight, —— = noon and --- = 6 o'clock. For days 30 & 31 **━━** = noon.)

The phase disc shows (local apparent solar) time of upper transit for each phase*. This time (and about 12½ hrs later) approximates the time of high tide at open ocean beaches. (It is delayed for bays and inlets.) Low tide is about 6 hours after high tide. Spring (highest) tides are at New and Full Moon, neap (moderate) tides at Half Moon.

Limitations

Line angles in the drawings are only accurate to the nearest whole degree and 2 lines must be aligned for a reading. The Moon moves away from the Sun at an average rate of about ½° per hour, so a 1° error in the Moon's position ≈ an error in time of ± 2 hrs. Allow a margin of ± 3 hrs when estimating the time of a lunar phase* with this instrument.

For better precision, calculate as follows: If the *first* New Moon is at time **t** am or **t** pm, the subsequent New Moons* will all be at time **t + n**, with am/pm of **t** alternating each time (i.e. am/pm of **t** *before* adding n will be the same as the first New Moon for all odd numbered New Moons, and the opposite for all even numbered New Moons.) Note: am/pm may change after adding n. For each New Moon, n is as follows:

2	0:44	**4**	2:12	**6**	3:40	**8**	5:08	**10**	6:36	**12**	8:05	**14**	9:33
3	1:28	**5**	2:56	**7**	4:24	**9**	5:52	**11**	7:21	**13**	8:49		

***The above calculation and this instrument predict phases based on the Moon's *average* motion. The time of the *real* phase may differ by up to ± 14½ hours, but often it is much closer than that.**

Notes

A lunation varies in length because the Moon's orbit is an ellipse, not a circle, so its orbital speed varies. But on average a lunar month is about 29½d, 44m long. Twelve of them ≈ 354 days, about 11 days shorter than a calendar year of 365d. Therefore, if the first New Moon of the year is before or on Jan 11, there will be a 13^{th} New Moon in Dec. (The 13^{th} lunation will end in January of next year.)

Something to exercise your powers of observation: In what way does this picture of the Moon differ from that shown in most (not all) textbooks?

Good resource for further reading and references: http://www.inconstantmoon.com
(I recommend that you begin navigation of this site by first clicking on "Lunar Tours" then "Phase Calc".)